Communication Theory II

Lecture 3: Review on Fourier analysis (cont’d)

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Section policy

- **Goal:** solve associated problems related to the topic of the lectures
- **Assistant lectures:** help you in solving the problem
  - Each student will solve the assigned questions and the assistant will help you if you have any questions
- **Needed materials:**

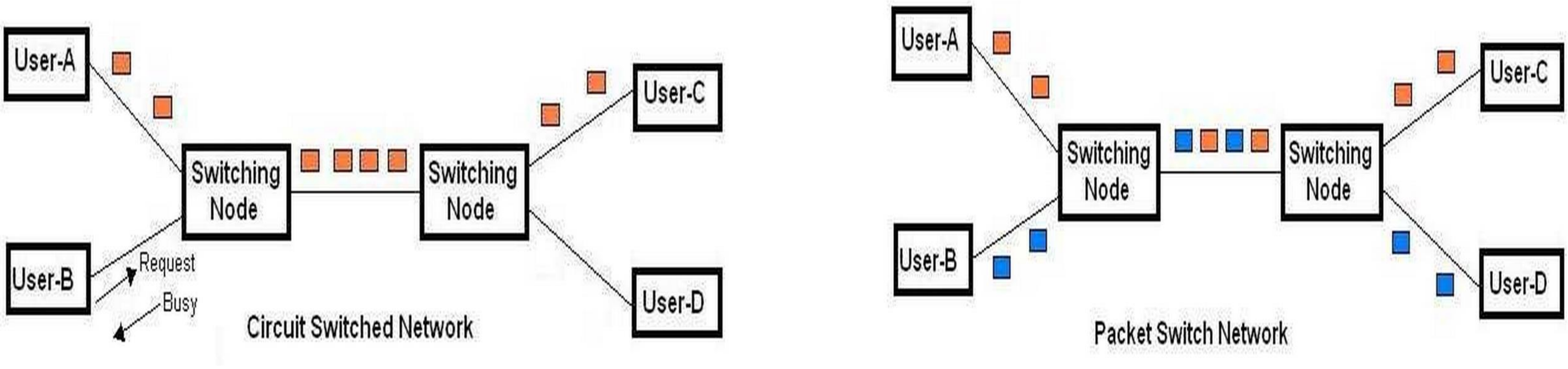
1) The sheet, Fourier tables (sofcopy or hardcopy)

2) A notebook, a pencil, and an eraser

Course Website

- **http://lms.mans.edu.eg/eng/**
- The site contains the lectures, quizzes, homework, and open forums for feedback and questions
- Log in using your name and password
- Password for quizzes: third

Circuit switching vs. packet switching



(a) (b)

Circuit switching (link sharing is based on fixed allocation) vs. packet switching (message is divided into segments or packets if necessary, sharing is per demand) *

* Image courtesy of *http://www.rfwireless-world.com/Terminology/circuit-switching-vs-packet-switching.html*

Lecture Outlines

- Review on Fourier analysis of signals and systems
    - The Fourier series
    - The Fourier transform
    - Relation between time and frequency representations
    - The Dirac delta function
    - Fourier transform of periodic signals
    - Transmission od signals through LTI systems
    - Hilbert transform

The Fourier series

- Let $g_{T_0}(t)$ denote a periodic signal, where the subscript T_0 denotes the duration of periodicity.
- By using a Fourier series expansion of this signal, we are able to resolve it into an infinite sum of sine and cosine terms:

$$\boxed{g_{T_0}(t) = a_0 + 2\sum_{n=1}^{\infty}[a_n\cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)]}$$

- Where $f_0 = \frac{1}{T_0}$ is the fundamental frequency
- a_n and b_n represent the amplitudes of the cosine and sine terms, respectively
- $\cos(2\pi n f_0 t)$ and $\sin(2\pi n f_0 t)$ are called basis functions

Basis functions ($\cos(2\pi n f_0 t)$ and $\sin(2\pi n f_0 t)$)

- Form an orthogonal set over the period T_0
- In that they satisfy three conditions:

$$\int_{-T_0/2}^{T_0/2} \cos(2\pi m f_0 t)\cos(2\pi n f_0 t)\,\mathrm{d}t = \begin{cases} T_0/2, & m = n \\ 0, & m \neq n \end{cases}$$

$$\int_{-T_0/2}^{T_0/2} \cos(2\pi m f_0 t)\sin(2\pi n f_0 t)\,\mathrm{d}t = 0, \qquad \text{for all } m \text{ and}$$

$$\int_{-T_0/2}^{T_0/2} \sin(2\pi m f_0 t)\sin(2\pi n f_0 t)\,\mathrm{d}t = \begin{cases} T_0/2, & m = n \\ 0, & m \neq n \end{cases}$$

Mean value (a_0) and other coefficients

- Integrate over a complete period both sides of the equation:

$$g_{T_0}(t) = a_0 + 2\sum_{n=1}^{\infty}[a_n\cos(2\pi n f_0 t) + b_n\sin(2\pi n f_0 t)]$$

yields: $$\boxed{a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{T_0}(t)\,\mathrm{d}t}$$

- Other coefficients:

$$\boxed{a_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{T_0}(t)\cos(2\pi n f_0 t)\,\mathrm{d}t, \qquad n = 1, 2, \ldots}$$

$$\boxed{b_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{T_0}(t)\sin(2\pi n f_0 t)\,\mathrm{d}t, \qquad n = 1, 2, \ldots}$$

Existence of Fourier series

- A periodic signal can be expanded in a Fourier series if the signal satisfies the *Dirichlet conditions*:

1. The function $g_{T_0}(t)$ is single valued within the interval T_0.
2. The function $g_{T_0}(t)$ has at most a finite number of discontinuities in the interval T_0.
3. The function $g_{T_0}(t)$ has a finite number of maxima and minima in the interval T_0.
4. The function $g_{T_0}(t)$ is absolutely integrable; that is,

$$\boxed{\int_{-T_0/2}^{T_0/2} \left|g_{T_0}(t)\right| \mathrm{d}t < \infty}$$

- From an engineering perspective, however, it suffices to say that the *Dirichlet conditions* are satisfied by the periodic signals encountered in communication systems

Complex exponential Fourier series

$$g_{T_0}(t) = a_0 + 2\sum_{n=1}^{\infty}[a_n\cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)]$$

$$\cos(2\pi n f_0 t) = \frac{1}{2}[\exp(\mathrm{j}2\pi n f_0 t) + \exp(-\mathrm{j}2\pi n f_0 t)]$$

$$\sin(2\pi n f_0 t) = \frac{1}{2\mathrm{j}}[\exp(\mathrm{j}2\pi n f_0 t) - \exp(-\mathrm{j}2\pi n f_0 t)]$$

$$g_{T_0}(t) = a_0 + \sum_{n=1}^{\infty}[(a_n - \mathrm{j}b_n)\exp(\mathrm{j}2\pi n f_0 t) + (a_n + \mathrm{j}b_n)\exp(-\mathrm{j}2\pi n f_0 t)]$$

Let c_n denote a complex coefficient related to a_n and b_n by

$$c_n = \begin{cases} a_n - \mathrm{j}b_n, & n > 0 \\ a_0, & n = 0 \\ a_n + \mathrm{j}b_n, & n < 0 \end{cases}$$

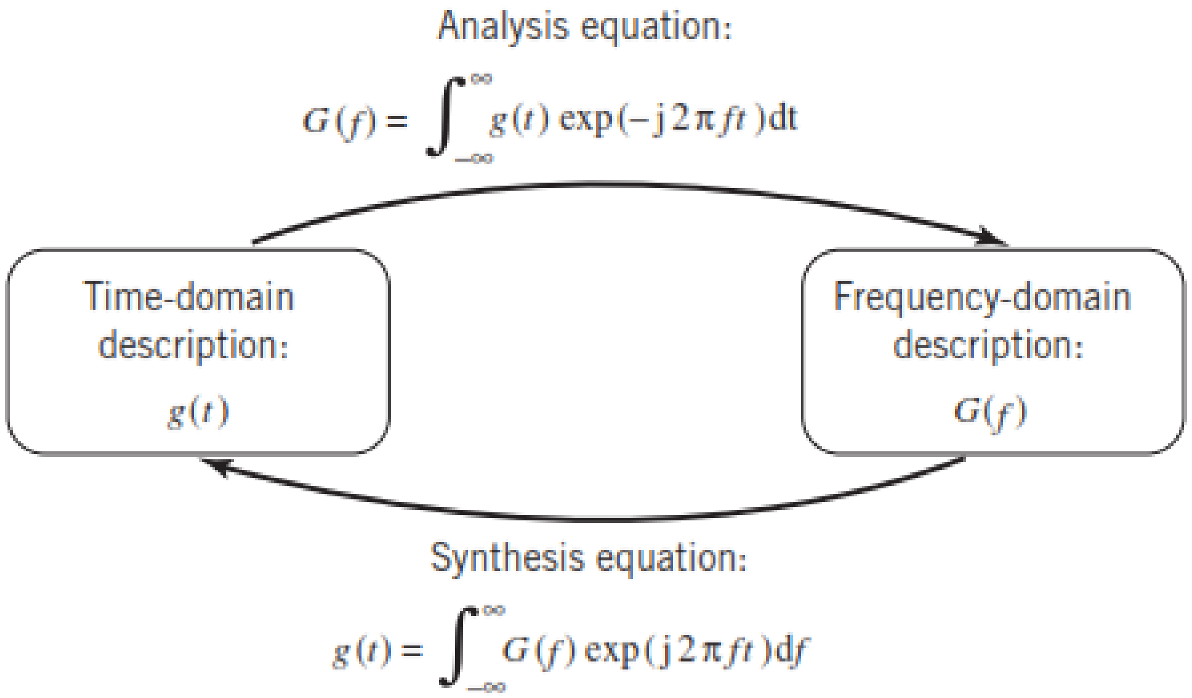
$$g_{T_0}(t) = \underbrace{\sum_{n=-\infty}^{\infty} c_n \exp(\mathrm{j}2\pi n f_0 t)}$$

Inner product of the signal with the basis function $[\exp(-\mathrm{j}2\pi n f_0 t)]$

Where $$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{T_0}(t)\exp(-\mathrm{j}2\pi n f_0 t)\,\mathrm{d}t, \qquad n = 0, \pm 1, \pm 2, \ldots$$

The Fourier Transform

- Representation of non-periodic signals
- Derivation is based on Fourier series (text book: page 16,17)



Analysis equation:

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt$$

Time-domain description:

$g(t)$

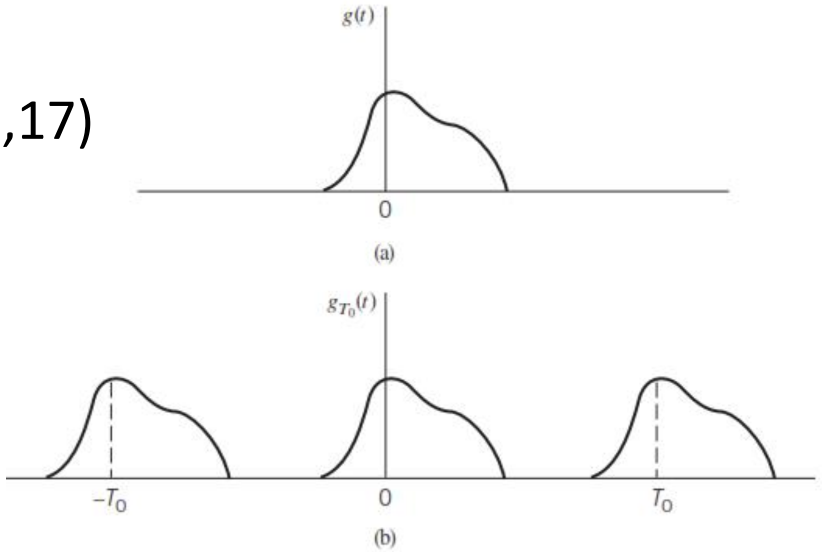
Frequency-domain description:

$G(f)$

Synthesis equation:

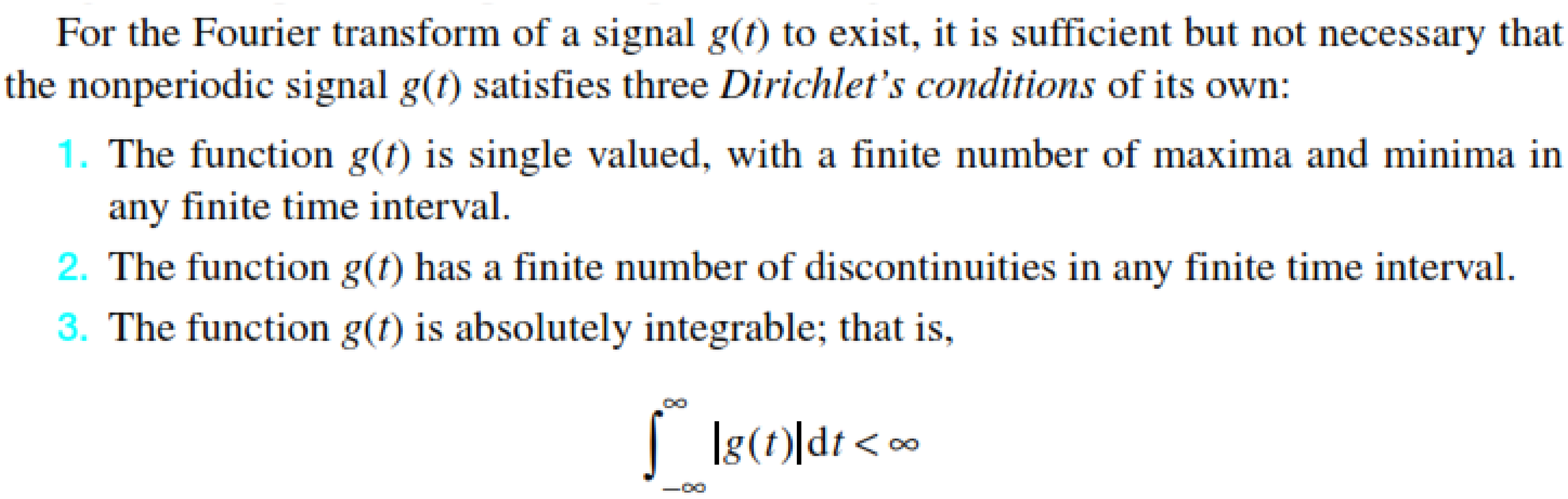
$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)df$$

$$g(t) = \lim_{T_0 \to \infty} g_{T_0}(t)$$



igure 2.1 Illustrating the use of an arbitrarily defined function of time to onstruct a periodic waveform. (a) Arbitrarily defined function of time $g(t)$. ›) Periodic waveform $g_{T_0}(t)$ based on $g(t)$.

Existence of Fourier Transform



For the Fourier transform of a signal $g(t)$ to exist, it is sufficient but not necessary that the nonperiodic signal $g(t)$ satisfies three *Dirichlet's conditions* of its own:

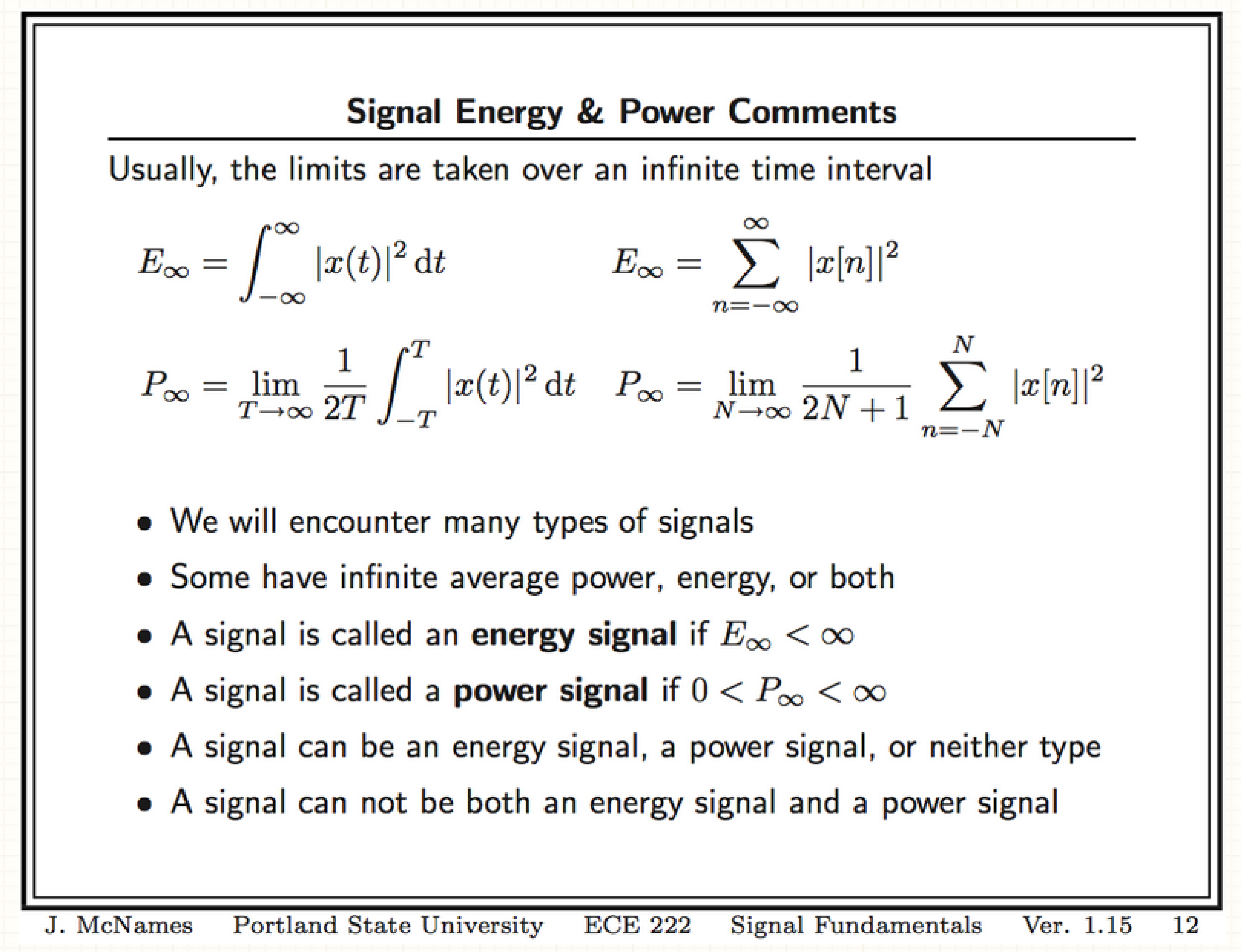
1. The function $g(t)$ is single valued, with a finite number of maxima and minima in any finite time interval.
2. The function $g(t)$ has a finite number of discontinuities in any finite time interval.
3. The function $g(t)$ is absolutely integrable; that is,

$$\int_{-\infty}^{\infty} |g(t)| \mathrm{d}t < \infty$$

- Physical realizability is a sufficient condition for the existence of a Fourier transform (e.g., all energy signals are Fourier transformable).

Condition for energy signal: $\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{d}t < \infty$

Energy vs. power signals



Signal Energy & Power Comments

Usually, the limits are taken over an infinite time interval

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t \qquad E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t \qquad P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- We will encounter many types of signals
- Some have infinite average power, energy, or both
- A signal is called an **energy signal** if $E_\infty < \infty$
- A signal is called a **power signal** if $0 < P_\infty < \infty$
- A signal can be an energy signal, a power signal, or neither type
- A signal can not be both an energy signal and a power signal

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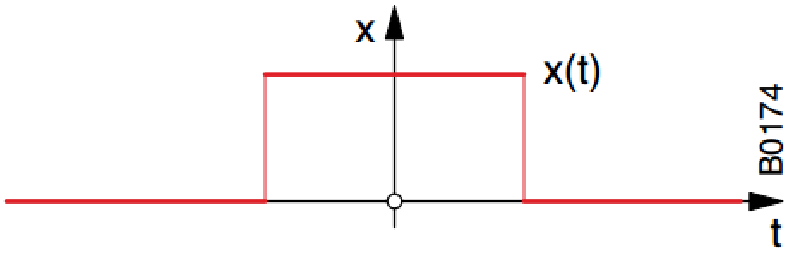
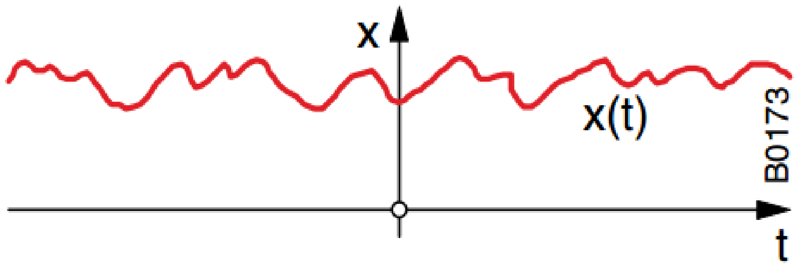
Example: Energy Signal



Example: Power Signal



Fourier Transform

- Provides mathematical tools for measuring the frequency content or spectrum of a signal
- $G(f)$ is the Fourier transform or the spectrum of the signal

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)\,dt$$

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)\,df$$

$$G(f) = \mathbf{F}[g(t)] \quad \rightleftharpoons \quad g(t) = \mathbf{F}^{-1}[G(f)]$$

$$g(t) \rightleftharpoons G(f)$$

- Simple evaluation of magnitude spectrum $|G(f)|$ and phase spectrum $\angle G(f)$

Tables

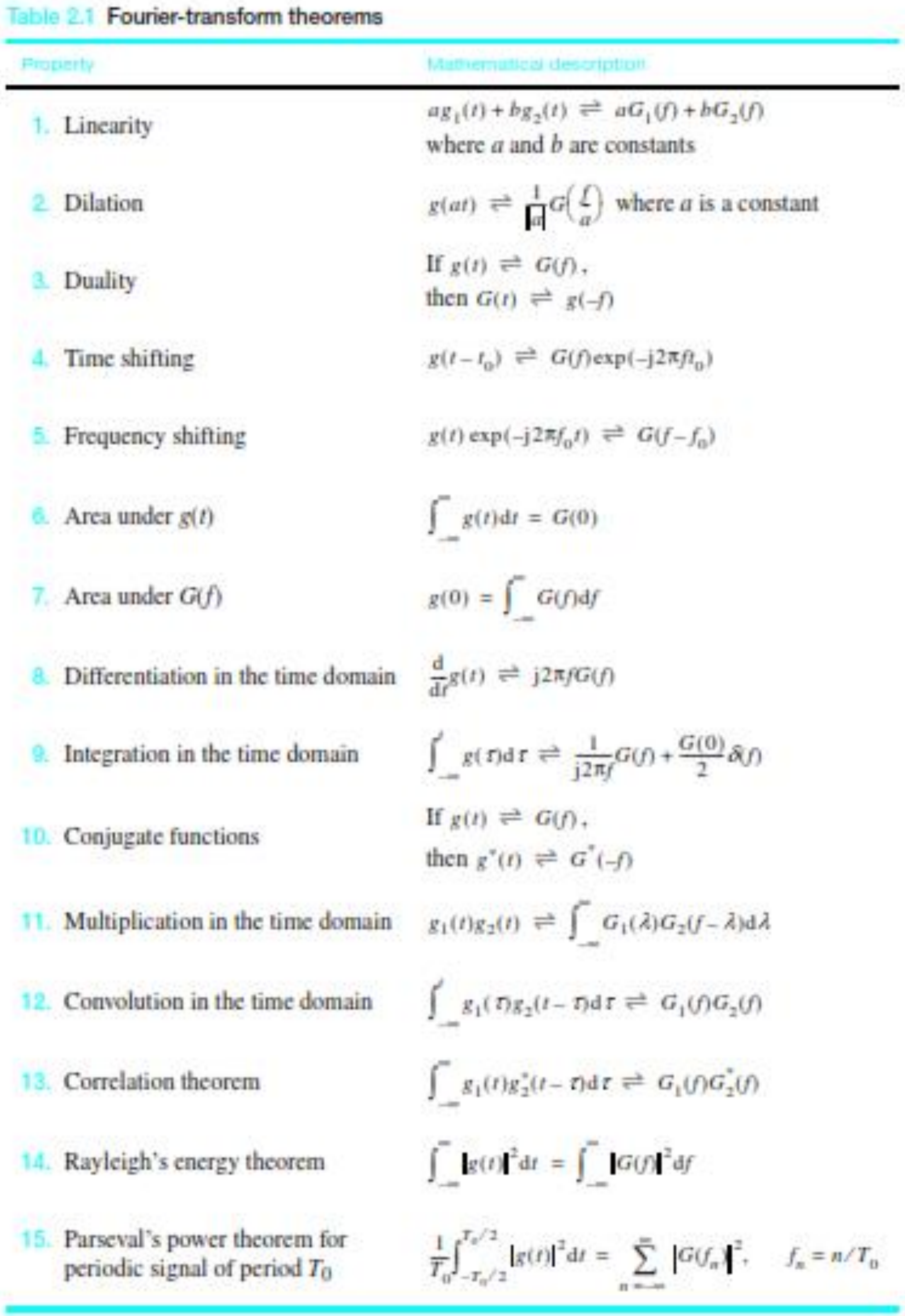


Table 2.1 Fourier-transform theorems

Property	Mathematical description
1. Linearity	$a g_1(t) + b g_2(t) \rightleftharpoons a G_1(f) + b G_2(f)$ where a and b are constants
2. Dilation	$g(at) \rightleftharpoons \frac{1}{\lvert a \rvert} G\left(\frac{f}{a}\right)$ where a is a constant
3. Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t - t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)$
5. Frequency shifting	$g(t)\exp(-j2\pi f_0 t) \rightleftharpoons G(f - f_0)$
6. Area under $g(t)$	$\int_{-\infty}^{\infty} g(t)\,dt = G(0)$
7. Area under $G(f)$	$g(0) = \int_{-\infty}^{\infty} G(f)\,df$
8. Differentiation in the time domain	$\frac{d}{dt} g(t) \rightleftharpoons j2\pi f G(f)$
9. Integration in the time domain	$\int_{-\infty}^{t} g(\tau)\,d\tau \rightleftharpoons \frac{1}{j2\pi f} G(f) + \frac{G(0)}{2}\delta(f)$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$
11. Multiplication in the time domain	$g_1(t) g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda) G_2(f - \lambda)\,d\lambda$
12. Convolution in the time domain	$\int_{-\infty}^{t} g_1(\tau) g_2(t - \tau)\,d\tau \rightleftharpoons G_1(f) G_2(f)$
13. Correlation theorem	$\int_{-\infty}^{\infty} g_1(t) g_2^*(t - \tau)\,d\tau \rightleftharpoons G_1(f) G_2^*(f)$
14. Rayleigh's energy theorem	$\int_{-\infty}^{\infty} \lvert g(t) \rvert^2\,dt = \int_{-\infty}^{\infty} \lvert G(f) \rvert^2\,df$
15. Parseval's power theorem for periodic signal of period T_0	$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \lvert g(t) \rvert^2\,dt = \sum_{n=-\infty}^{\infty} \lvert G(f_n) \rvert^2, \quad f_n = n/T_0$

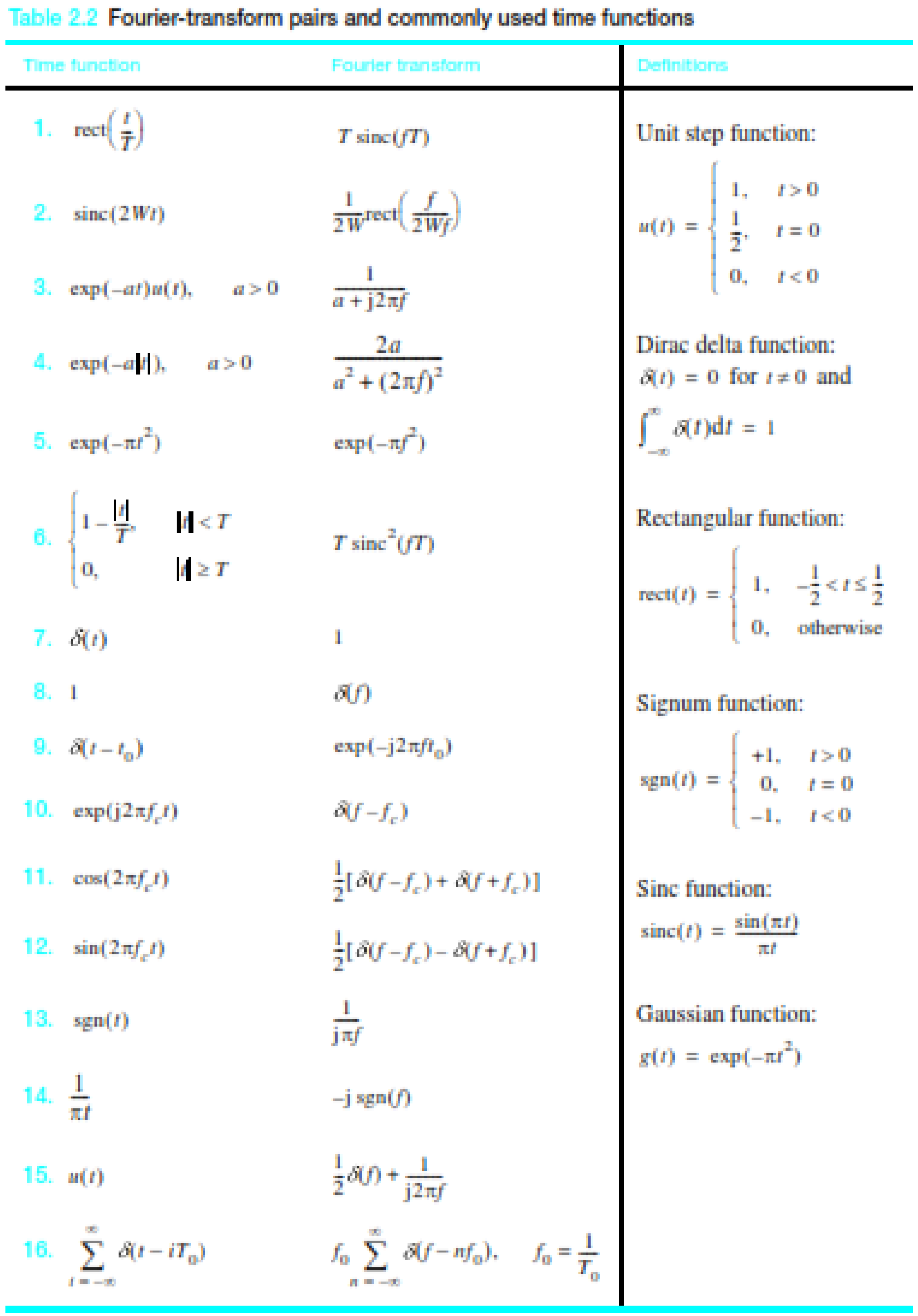


Table 2.2 Fourier-transform pairs and commonly used time functions

	Time function	Fourier transform
1.	$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}(fT)$
2.	$\mathrm{sinc}(2Wt)$	$\frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$
3.	$\exp(-at)u(t), \quad a>0$	$\frac{1}{a+\mathrm{j}2\pi f}$
4.	$\exp(-a\lvert t\rvert), \quad a>0$	$\frac{2a}{a^2+(2\pi f)^2}$
5.	$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
6.	$\begin{cases} 1-\frac{\lvert t\rvert}{T}, & \lvert t\rvert < T \\ 0, & \lvert t\rvert \geq T \end{cases}$	$T\,\mathrm{sinc}^2(fT)$
7.	$\delta(t)$	1
8.	1	$\delta(f)$
9.	$\delta(t-t_0)$	$\exp(-\mathrm{j}2\pi f t_0)$
10.	$\exp(\mathrm{j}2\pi f_c t)$	$\delta(f-f_c)$
11.	$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$
12.	$\sin(2\pi f_c t)$	$\frac{1}{2\mathrm{j}}[\delta(f-f_c)-\delta(f+f_c)]$
13.	$\mathrm{sgn}(t)$	$\frac{1}{\mathrm{j}\pi f}$
14.	$\frac{1}{\pi t}$	$-\mathrm{j}\,\mathrm{sgn}(f)$
15.	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{\mathrm{j}2\pi f}$
16.	$\sum_{i=-\infty}^{\infty}\delta(t-iT_0)$	$f_0\sum_{n=-\infty}^{\infty}\delta(f-nf_0), \quad f_0=\frac{1}{T_0}$

Definitions

Unit step function:

$$u(t) = \begin{cases} 1, & t>0 \\ \frac{1}{2}, & t=0 \\ 0, & t<0 \end{cases}$$

Dirac delta function:
$\delta(t) = 0$ for $t \neq 0$ and

$$\int_{-\infty}^{\infty}\delta(t)\,\mathrm{d}t = 1$$

Rectangular function:

$$\mathrm{rect}(t) = \begin{cases} 1, & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Signum function:

$$\mathrm{sgn}(t) = \begin{cases} +1, & t>0 \\ 0, & t=0 \\ -1, & t<0 \end{cases}$$

Sinc function:

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Gaussian function:

$$g(t) = \exp(-\pi t^2)$$

Relation between time and frequency response

- If the time-domain description of a signal is changed, the frequency-domain description of the signal is changed in an inverse manner, and vice versa.

- If a signal is strictly limited in frequency, then the time-domain description of the signal will trail on indefinitely, even though its amplitude may assume a progressively smaller value.

A signal is strictly limited in frequency (i.e., strictly band limited) if its Fourier transform is exactly zero outside a finite band of frequencies.

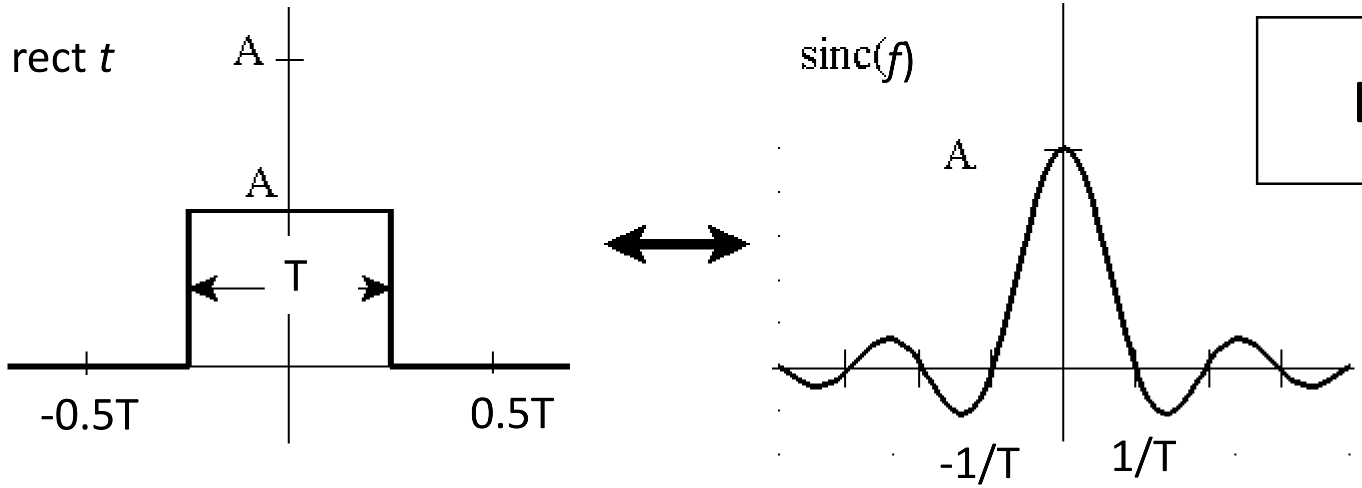
- If a signal is strictly limited in time (i.e., the signal is exactly zero outside a finite time interval), then the spectrum of the signal is infinite in extent, even though the magnitude spectrum may assume a progressively smaller value.

- A signal cannot be strictly limited in both time and frequency

- **Time bandwidth product:** duration × bandwidth = constant

The bandwidth dilemma

- The bandwidth of a signal provides a measure of the extent of significant spectral content of the signal for positive frequencies.
- When the signal is strictly band limited, the bandwidth is well defined. For example, the sinc pulse sinc(2W*t*) has a bandwidth equal to W.
- If a signal is low-pass (i.e., its spectral content is centered around the origin $f = 0$), *the bandwidth is defined as one-half the total width of the main spectral* lobe, since only one-half of this lobe lies inside the positive frequency region

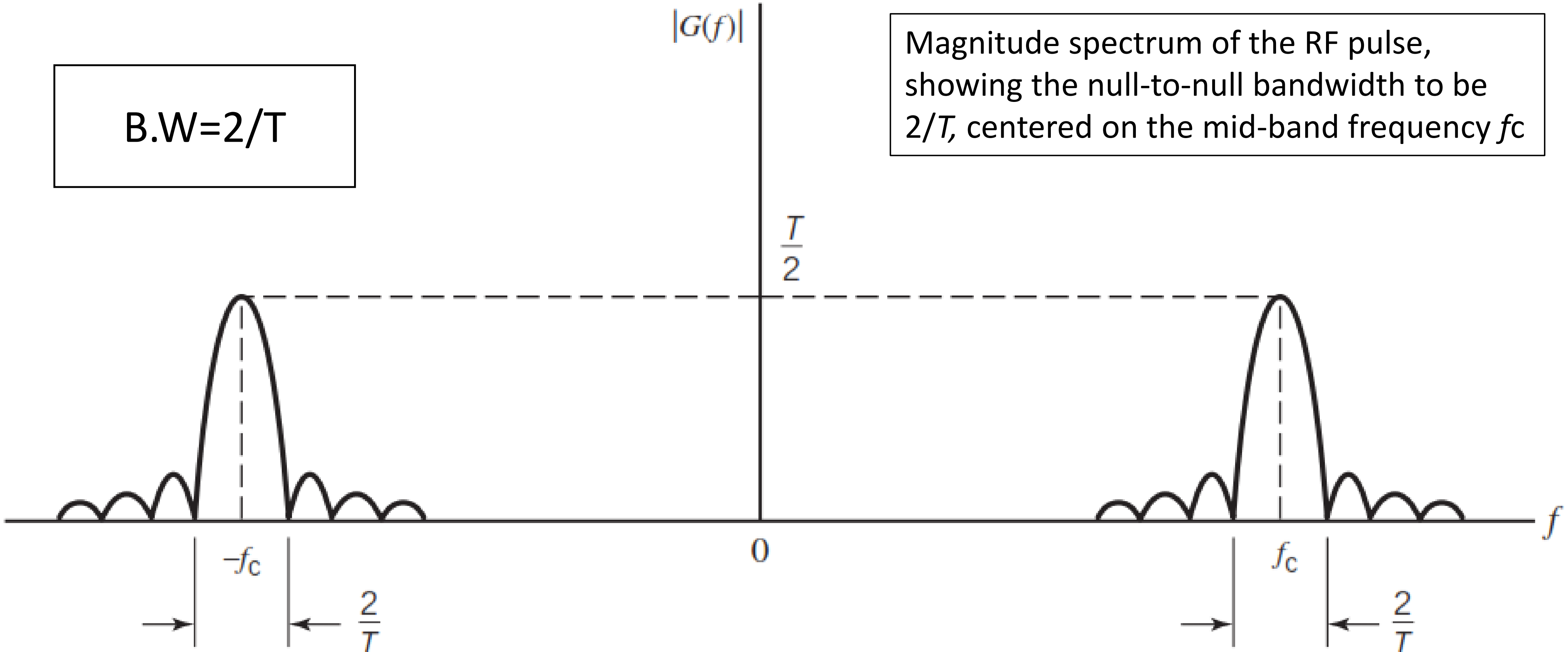


B.W=1/T

The bandwidth dilemma (con’d)

- If the signal is *band-pass with main spectral lobes centered around* ±*fc* where *fc* is large enough, the bandwidth is defined as the width of the main lobe for positive frequencies. This definition of bandwidth is called the *null-to-null bandwidth.*

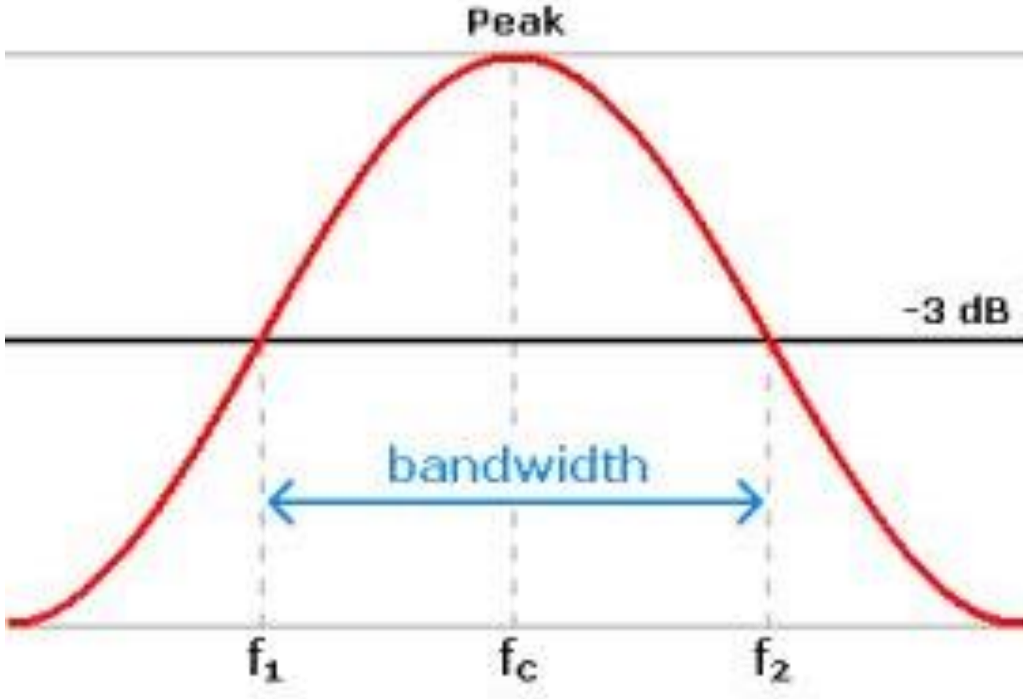
B.W=2/T



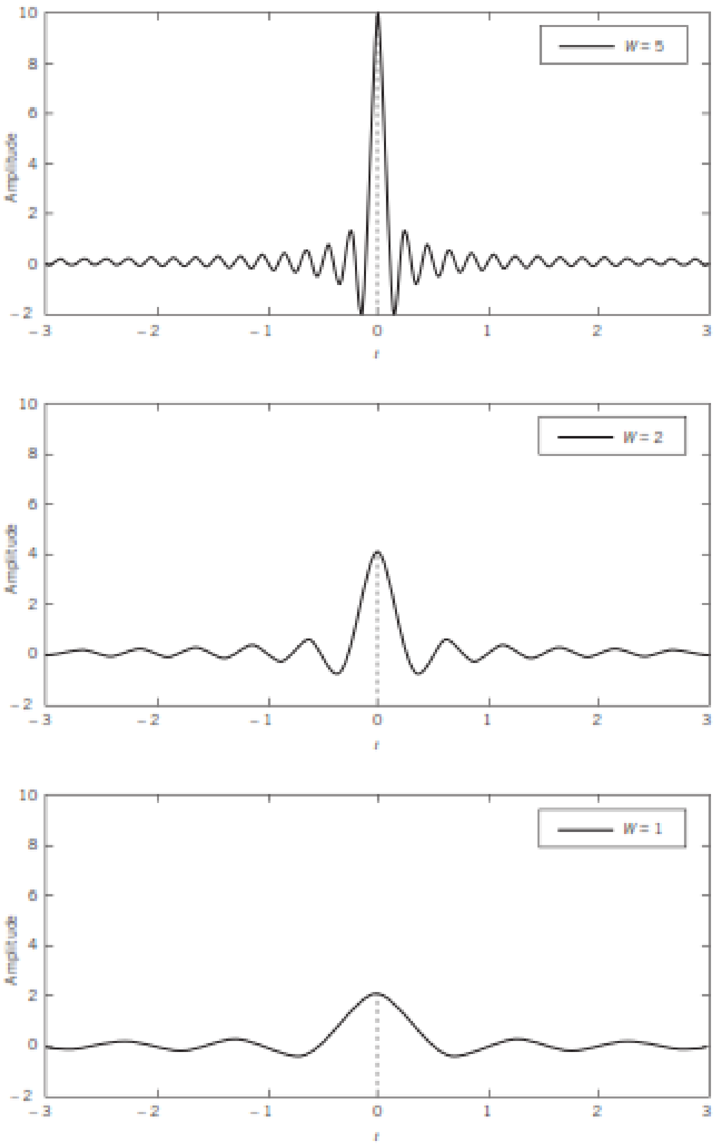
Magnitude spectrum of the RF pulse, showing the null-to-null bandwidth to be 2/*T*, centered on the mid-band frequency *fc*

3 dB bandwidth

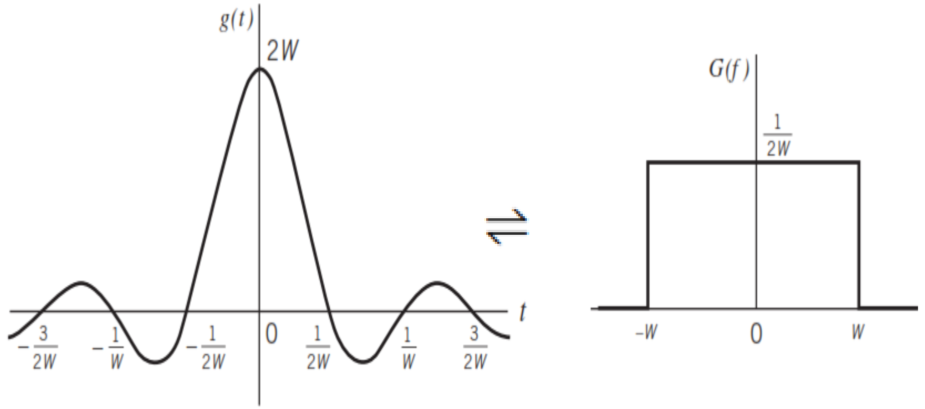
The 3 dB bandwidth of a **low-pass signal** is defined as the separation between zero frequency, where the magnitude spectrum attains its peak value, and the positive frequency at which the amplitude spectrum drops to of its peak value.



The Dirac delta function



Evolution of the sinc function $2W$ *sinc(*$2Wt$*) toward the delta function as the* parameter W *progressively increases*



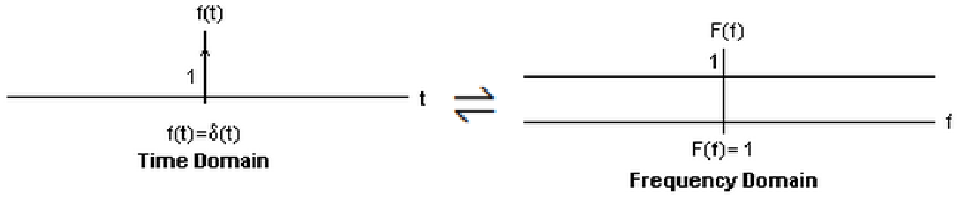
$$\mathrm{sinc}(2Wt) \rightleftharpoons \frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$$

$$\downarrow$$

$$\boxed{\delta(t) = \lim_{W \to \infty} 2W\,\mathrm{sinc}(2Wt)}$$

$$\downarrow$$

$$\delta(t) \rightleftharpoons 1$$



Dirac delta function:

$\delta(t) = 0$ for $t \neq 0$ and

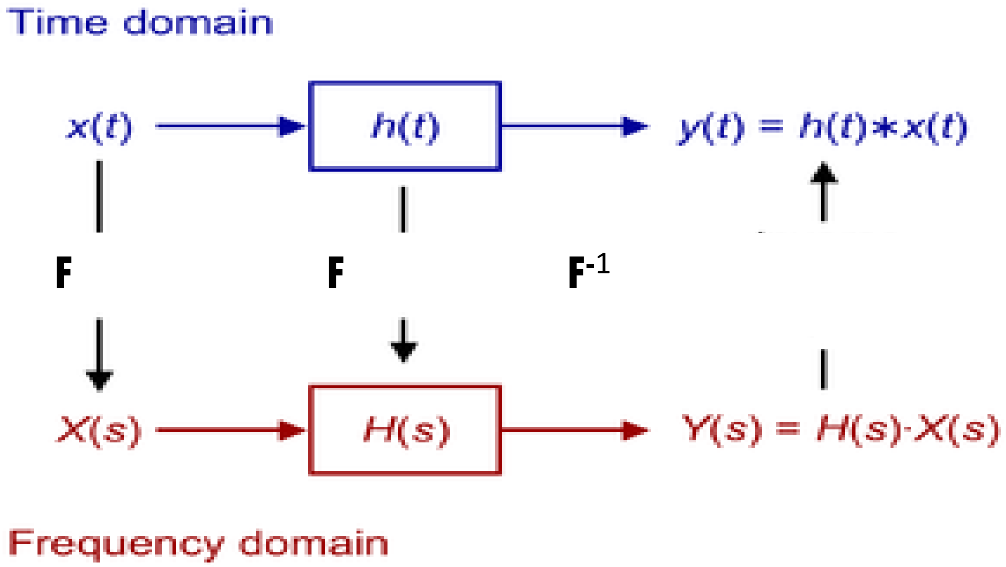
$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1$$

Fourier transform of periodic signals

- Can Fourier transform works for periodic signals?

Fourier transform for analyzing signals and systems

- Provides the mathematical link between the time domain of a signal (waveform) and its frequency domain (spectrum)
- Time and frequency response of a linear time-invariant (LTI) system defined in terms of its impulse response and frequency response, respectively



LTI system

If $x_1(n) \rightarrow \boxed{h(n)} \rightarrow y_1(n)$

and $x_2(n) \rightarrow \boxed{h(n)} \rightarrow y_2(n)$

Then $a_1y_1(n)+a_2y_2(n)=H\{a_1x_1(n)+a_2x_2(n)\}$

Where a1, a2 are scalars

(a)

If $x_1(n) \rightarrow \boxed{h(n)} \rightarrow y_1(n)$

Then $x_1(n\text{-}k) \rightarrow \boxed{h(n)} \rightarrow y_1(n\text{-}k)$

(b)

A (a) linear and (b) a time invariant system

Questions